Traces of Torsion Units

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Abstract

A conjecture due to Zassenhaus asserts that if G is a finite group then any torsion unit in $\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to an element of G. Here a weaker form of this conjecture is proved for some infinite groups.

1 Introduction

Let G be a group and let $\mathcal{U}_1(\mathbb{Z}G)$ be the group of units of augmentation one of the integral group ring $\mathbb{Z}G$. One of the ways to study elements of $\mathcal{U}_1(\mathbb{Z}G)$ is to study certain traces one of which is the augmentation. We recall some of these traces which not only have an important role in the study of units but are also objects of important conjectures in the field. Given elements $\alpha = \sum \alpha(g)g \in \mathbb{Z}G$ and $g_0 \in G$, let $n = o(g_0)$ and denote by C_{g_0} the conjugacy class of g_0 in G. Then $\tilde{\alpha}(g_0) = \sum_{h \in C_{g_0}} \alpha(h)$ and $T^n(\alpha) = \sum_{o(g)=n} \alpha(g)$ are well known traces. The support of α, $supp(\alpha)$, is the union of the conjugacy classes for which $\tilde{\alpha}(g) \neq 0$. Several important conjectures are related to these traces. A conjecture of A.A. Bovdi states that $T^n(\alpha)$ is always non-negative (see [1]). This conjecture was proved to be true for supersoluble and Frobenius groups by S.O. Juriaans (see [11]) and for metabelian groups by M. Dokuchaev and S.K. Sehgal (see [7]). Actually, in [7], Dokuchaev and Sehgal defined more general traces and showed that these traces are also always non-negative for abelian-by-polycyclic groups and for metabelian groups. As a consequence, they confirmed Bovdi's conjecture for metabelian groups. In general, Bovdi's conjecture is still open and no counterexamples are known.

Another major problems in the field is the description of the unit group of the integral group ring of a finite group, i.e., give a presentation of the unit group. This has been done in only some exceptional cases (see [10, 12, 18]). An apparently more easy problem is to study the units of finite order. Let α be a unit in $\mathcal{U}_1(\mathbb{Z}G)$ of finite order. In case G is finite, some interesting relations exist between the order of α and certain elements of G, e.g., if the order of $g_0 \in G$ and α are coprime then $\tilde{\alpha}(g_0) = 0$. Other results on traces were also established by A.A. Bovdi and many other researchers in the field. For example, if G is a finite group then a conjecture of Zassenhaus (see [18, 19, 20]) states that every torsion element $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ is rationally conjugate to a group element (see [15]). S. Berman, Polcino Milies, J. Ritter, S.K. Sehgal, E. Jespers, M. Mazur, S.O. Juriaans, A. Weiss, L. Scott, K.W. Roggenkamp, A. del Rio, M. Hertweck, I.B.S. Passi, A.A. Bovdi, Kimmerle, among many other researchers in the field, contributed to partial solutions of this conjecture.

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For finite groups this is equivalent to the following (see [2, 19]): for every $\gamma \in \langle \alpha \rangle$ there exists an element $g_0 \in G$, unique up to conjugacy, such that $\tilde{\gamma}(g_0) \neq 0$. This equivalent form of Zassenhaus' conjecture was used to confirm this conjecture. One of the first papers in this direction is due to Passi-Luthar-Trama ([14]) where they confirm this conjecture for the group A_5. In [6], this method was also use to confirm a stronger conjecture, also due to Zassenhaus, for nilpotent-by-nilpotent groups. However, this is not valid anymore if we consider G to be infinite. In general, technics from representation theory fail in this case. In [7] interesting results on traces were obtained for important classes of infinite groups. Bovdi, Marciniack and S.K. Seghal proved (see [2]) the following nice result: Let G be an infinite group, α a torsion unit in $\mathcal{U}_1(\mathbb{Z}G)$ and $g_0 \in G$ of infinite order; then $\tilde{\alpha}(g_0) = 0$. This is a key result which permits one to extend several results of finite groups to infinite groups. For example, it is shown that traces of nilpotent groups are non-negative. However for finitely generated infinite nilpotent groups a counterexample to Zassenhaus' conjecture is known (see [13]) showing that rational conjugacy is somehow too strong if G is infinite. This leads to the following definition: A unit $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ is said to have the *unique trace property* if there exists an element $g \in G$, unique up to conjugacy, such that $\tilde{\alpha}(g) \neq 0$.

As in [2], a group G has the *unique trace property* (**UT**-*property*) if every element $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ of finite order has the unique trace property. In [2] it is proved that nilpotent groups are UT-groups which can be considered a weak form of Zassenhaus' conjecture which cannot be generalized since, as mentioned above, it is known that there are nilpotent groups having torsion units which are not rationally conjugate to a group element (see [13]). If p is a rational prime, we say that a group G is a **p-UT** group if every torsion unit of prime power order has the unique trace property.

In this paper we continue the study of traces for infinite groups. As a counterexample to the Zassenhaus Conjecture is known for infinite groups this makes the study of traces interesting on its own. We concentrate on those related to the UT-property. We start with some preliminary results based on [2]. These are then used to exhibit new classes of UT and p-UT groups. The main difficult is to show that if G is a group and $\alpha \in \mathbb{Z}G$ is a torsion unit, then $\tilde{\alpha}(g) = 0$ for elements $g \in G$ of infinite order. Using the technic of reduction to the finite case, we manage to prove the UT-property to some infinite groups. Along the way we obtain some results that might be of independent interest. For example, we prove that for polycyclic-by-finite groups the order of torsion subgroups of the integral unit group is bounded.

2 Preliminary Results

We begin by proving a result which, in some cases, deals with the traces of elements of infinite order. This is a slight generalization of a result due to Bovdi, Marciniak and Sehgal. As a consequence we show that if G is polycyclic-by-finite then the size of the torsion subgroup of $\mathbb{Z}G$ has an upper bound.

Proposition 2.1 *Let G be a locally noetherian by finite group and $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ a torsion unit. Then $\tilde{\alpha}(g) = 0$ for any $g \in G$ of infinite order.*

Proof. Suppose that $\tilde{\alpha}(g) \neq 0$. Then, by [2, Prop. 2] there exists an integer $k > 1$ and an element $x \in G$ such that $x^{-1}gx = g^k$. If x is of finite order, set $m = o(x)$. Then $g = x^{-m}gx^m = g^{k^m}$ and hence we have a contradiction.

Suppose that $o(x) = \infty$ and let H be a normal locally noetherian subgroup of finite index in G. Then there exists an integer $m > 0$ such that x^m *and* g^m are in H. Set $t = x^m,\ h = g^m,\ n = k^m$

and $H_0 = \langle t, h \rangle$. Then $t^{-1}ht = h^n$, and since H_0 is noetherian we must have that $n = 1$ and consequently $k = 1$, a contradiction. ∎

The proof of the proposition does not require H to be normal. The only thing needed is that for any $g \in G$ of infinite order, there exists an integer $n = n(g)$ such that $g^n \in H$. As a consequence we have the following result.

Corolarry 2.2 *Let G be a group, H a normal locally noetherian torsion free subgroup of finite index in G and $A < \mathcal{U}_1(\mathbb{Z}G)$ a finite subgroup. Then the order of A divides $[G : H]$.*

Proof. If we denote by Ψ the natural projection of $\mathbb{Z}G$ onto $\mathbb{Z}(G/H)$ then we just have to show that Ψ is injective on A. In order to show this let α be an element of A which is mapped to 1 by Ψ. We have $1 = \Psi(\alpha) = \sum_{g \notin H} \alpha_g \Psi(g) + \sum_{g \in H} \alpha_g \Psi(g) = \sum_{g \notin H} \alpha_g \Psi(g) + \alpha(1)$. In the last equality we used the torsion freeness of H and Proposition 2.1. It follows that $1 \in supp(\alpha)$ and thus, by [16, Theorem 7.3.1], $\alpha = 1$. ∎

The infinite dihedral group has a normal cyclic subgroup of index 2. Hence a non-trivial finite subgroup of $\mathcal{U}_1(\mathbb{Z}G)$ has order 2 (see [18, Theorem 46.9]).

Given a group G, $T(G)$ denotes the set of elements of finite order of G. In general this is not a subgroup of G.

3 The UT and p-UT Property

In this section we study the UT and p-UT-property and give examples of classes of groups having one of these properties. As mentioned in the introduction, for finite group this is equivalent to the Zassenhaus Conjecture or a p-version of this conjecture. To prove our results we use some interesting ideas from [4, 5, 7] together with a reduction to finite quotients. This enables us to lift results on traces for integral group rings of finite groups to integral group rings of some classes of infinite groups.

Theorem 3.1 *Let G be a group, H a normal locally noetherian torsion free subgroup of finite index and suppose that $T(G)$ is a subgroup. If G/H is a UT-group then G is a also a UT-group.*

Proof. Let $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ be a torsion unit and $g \in G$ an element. If g is of infinite order then, by Proposition 2.1, we have that $\tilde{\alpha}(g) = 0$.

If g has finite order, denote by β and $\overline{g}$ the projections of α and g in $\mathcal{U}_1(\mathbb{Z}(G/H))$. Let $C_{\overline{g}}$ be the conjugacy class of $\overline{g}$. Then it is easy to see that $C_{\overline{g}}$ is the projection of the subset $S = \{k \in G : k = t^{-1}gth,\ h \in H,\ t \in G\}$. Since $T(G)$ is a normal subgroup and H is normal and torsion free, we see that $S \cap T(G) = C_g$. Furthermore, if we write $S = S_1 \cup C_g$, where S_1 are the elements of infinite order of S, then S_1 is a normal subset of G. Writing S_1 as a disjoint union of conjugacy classes and applying Proposition 2.1, it follows that $\sum_{h \in S_1} \alpha(h) = 0$ and hence $\tilde{\beta}(\overline{g}) = \sum_{h \in S} \alpha(h) = \tilde{\alpha}(g)$. Since, by our assumption, G/H is a UT-group the result follows. ∎

Corolarry 3.2 ([2]) *Let G be a locally nilpotent group. Then G is a UT-group.*

Proof. We may suppose that G is finitely generated. This, together with [17, 5.4.6], [17, 5.4.15] and [20], gives that the hypothesis of the theorem are satisfied. Hence G is a UT-group. ∎

If G is a group and $g \in G$ is an element we denote by $K(g) = [g, G]$. Let now G be a group generated by an element t and an abelian normal subgroup A such that $t^{-1}at = a^{-1}$ for any $a \in A$ and $t^2 \in A$. Let $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ be a torsion unit. By Proposition 2.1, we have that $\tilde{\alpha}(g) = 0$ for every element of infinite order. Now let $g = ta \in G$ be an element which is not in A. We compute $K(g)$. If $b \in A$ then $[g, b] = [t, b] = b^{-2}$. If $h = tb$ then $[g, h] = [ta, tb] = [tb, t][a, tb] = [b, t][a, t] = (ba)^{-2}$. Hence $K(g) = \{a^2 : a \in A\}$. So we have the following result:

Lemma 3.3 *Let G be a group generated by an abelian subgroup A and t an element of G, such that $t^{-1}at = a^{-1}$ for any $a \in A$ and $t^2 \in A$. Then*

1. *For every $g \notin A$ we have that $K(g) = \{a^2 : a \in A\}$*
2. *If $g \notin A$ then $gK(g) = C_g$.*

Proof. The considerations above show that (1) holds. So, let $g\theta \in gK(g)$. Since $g \notin A$ we have that conjugation by g inverts the elements of A. By (1) we have that $\theta = \varphi^2$ for some $\varphi \in A$. Setting $t = g\varphi$ we see easily that $t^{-1}g\theta t = g$. ∎

Remark : Note that item (2) of the previous Lemma holds whenever the elements of $K(g)$ are squares and are inverted by g.

Recently, it was proved that the Zassenhaus conjecture is true for the class of finite groups $H = KX$, where X is a cyclic group which is normal in G and K is Abelian, [3, 8]. It would be valuable to know if the conjecture still holds for groups $G = KX$, where K is a normal Abelian subgroup of G and X is a cyclic subgroup.

Theorem 3.4 *Let $G = \langle t,\ A : t^2 \in A,\ t^{-1}at = a^{-1}\ ,\ \forall a \in A\rangle$ where A is an abelian normal subgroup of G. Then G is a p-UT-group.*

Proof. Let $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ be a torsion unit and $g \in G$ an element of the support of α. Suppose first that $o(g) = \infty$. Note that $[G : A] = 2$ and hence, by Proposition 2.1, $\tilde{\alpha}(g) = 0$. Secondly, suppose that $g \notin A$. By Lemma 3.3, we have that $gK(g) = C_g$. Notice that $gK(g)$ is central in the quotient group $G/K(g)$ and hence, by [2, Prop. 4], we have that $\tilde{\alpha}(g) = \sum_{h \in gK(g)} \alpha(g) = 0$ *or* 1.

Finally we consider a torsion element $g \in T(A)$; since the support of α is finite and $t^2 \in A$, we may suppose that A is finitely generated. In particular A is a polycyclic group and hence, by [17, 5.4.15], we have that there exist $H \lhd A$, which is torsion free and of finite index. Note that, since A is abelian and conjugation by t inverts the elements of A, H will also be normal in G. Consider the quotient group $\overline{G} = G/H$. The group $\overline{G}$ is metabelian and thus, by a result of [6], it has the p-UT-property. Let $\overline{g}$ be the projection of g in $\overline{G}$. Then it is easily seen that $C_{\overline{g}}$ is the projection of the subset $S = \{b \in A : b = x^{-1}axh,\ h \in H,\ x \in G\}$. Note that we may write S as a disjoint union $S = C_g \cup S_1$ where $S_1 = \{b \in S : h \neq 1\}$ is a normal subset of G whose elements are all of infinite order. Writing S_1 as a disjoint union of conjugacy classes, we conclude, by Proposition 2.1, that $\sum_{h \in S} \alpha(h) = \tilde{\alpha}(g)$. Consider the projection $\Psi : \mathbb{Z}G \longrightarrow \mathbb{Z}\overline{G}$ and let $\beta = \Psi(\alpha)$. Then, since $\overline{G}$ is a p-UT-group, we have that $\sum_{h \in S} \alpha(h) = \sum_{\overline{h} \in C_{\overline{g}}} \beta(\overline{h}) = \tilde{\beta}(\overline{g}) \in \{0, 1\}$. Hence $\tilde{\alpha}(g) \in \{0, 1\}$ for every element $g \in G$. Since α has augmentation 1, it follows that G has the p-UT property. ∎

A group G is called a T-group if normality is transitive in G. Let G be a solvable T-group and set $A = C_G(G')$. If A is not a torsion group then, by a result of [17, 13.4.9], we have that G satisfies the condition of Theorem 3.4 and hence G is a p-UT-group.

We now consider groups G whose derived subgroup is cyclic of infinite order, say $G' = \langle \rho \rangle$. We shall use this notation in the following results.

Lemma 3.5 *Let G be a group with cyclic derived subgroup; then*

1. *If $g \in T(G)$ centralizes ρ then g is central.*
2. *Elements of odd order are central.*
3. *$\{g^2 : g \in T(G)\} \subseteq \mathcal{Z}(G)$.*
4. *If $g \in G$ has infinite order and $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ is an element of finite order then $\tilde{\alpha}(g) = 0$.*

Proof.(1) Let $g \in T(G)$ and $x \in G$ then, since $\langle \rho \rangle$ is normal in G, we have that $g^{-1}xg = x\rho^k$ for some integer k. Let $m = o(g)$ then we have that $x = g^{-m}xg^m = x\rho^{km}$. Since ρ has infinite order we must have that $k = 0$.

(2) If $g \in G$ then g^2 centralizes ρ and hence is central. Since g has odd order we have that g is central.

(3) The proof of (2) applies.

(4) Suppose that this is false; then, by [2, Prop. 2], there exist $k > 1$, $x \in G$ such that $x^{-1}gx = g^k$. This implies that $g^{k-1} = [g, x] \in G'$. Set $n = k - 1$ *and* $h = g^n$; then the subgroup $\langle h \rangle$ is normal in G. Hence $x^{-1}hx \in \{h, h^{-1}\}$. But on the other hand $x^{-1}hx = h^k$ and hence we must have that $k = 1$, a contradiction. ∎

Lemma 3.6 *Let G be a group such that G' is infinite cyclic. Then, for any torsion element $g \in G$, we have that $gK(g) = C_g$.*

Proof. Let $G' = \langle \rho \rangle$. Then, since G' is a normal subgroup, we have that $g^{-1}\rho g \in \{\rho, \rho^{-1}\}$. If $g^{-1}\rho g = \rho$ then, by Lemma 3.5, g is central. So we may suppose that $g^{-1}\rho g = \rho^{-1}$. In this case also $g\rho g^{-1} = \rho^{-1}$. Hence we have that $g^{-1}g\rho^{-1}g = g\rho$, i.e., $g\rho^{-1}$ is conjugated to $g\rho$. We now separate the proof in two cases:

Case 1: $K(g) \neq G'$.

Since $g^{-1}\rho g = \rho^{-1}$ and $K(g)$ is cyclic we must have that $K(g) = \langle \rho^2 \rangle$. Hence, by the Remark following Lemma 3.3, we have that $gK(g) = C_g$.

Case 2: $K(g) = G'$.

In this case, since G' is cyclic and ρ is inverted by elements not in its centralizer, we see easily that there is an element $t \in G$ such that $K(g) = \langle [g, t] \rangle$. In particular, we have that $[g, t] \in \{\rho, \rho^{-1}\}$. Hence g is conjugated either to $g\rho$ *or to* $g\rho^{-1}$. Since we have already proved that $g\rho$ is conjugate to $g\rho^{-1}$, we only have to prove that an element of $gK(g)$ is either conjugate to g or to $g\rho$. In fact, set $h = g\theta$ with $\theta \in K(g)$. If θ were a square then, by the Remark following Lemma 3.3, h is conjugate to g. If θ is not a square, we may write $h = g\rho\varphi$ where φ is a square. Hence, again by the same Remark, we have that h is conjugated to $g\rho$ which in turn is conjugated to g. ∎

Theorem 3.7 *Let G be a group such that the derived subgroup of G is infinite cyclic. Then G is a UT-group.*

Proof. Let $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ be a torsion unit and $g \in G$ an element. If g is of infinite order then, by Proposition 2.1, we have that $\tilde{\alpha}(g) = 0$. If g is a torsion element then, by Lemma 3.6, we have that $\tilde{\alpha}(g) = \sum_{h \in gK(g)} \alpha(h)$. Since the element $gK(g)$ is central in the quotient group $G/K(g)$ we have, by [2, Prop. 4], that $\sum_{h \in gK(g)} \alpha(h) \in \{0, 1\}$. Since α has augmentation 1, the result is proved. ∎

Let G be a group and (A_n) a descending chain of normal subgroups of G. Denote by $\Psi_n : G \longrightarrow G/A_n$ the natural map and let F_n be the pre-image of $\Psi_n(C_{g_0})$, with $g_0 \in G$. In what follows we shall use this notation.

Proposition 3.8 *Let G be a group and (A_n) a descending chain of normal subgroups of G such $\bigcap A_n = 1$. Then, with the notation above, for every element $g_0 \in G$ we have that $\bigcap F_n = C_{g_0}$.*

Proof. Let $\Psi_n : G \longrightarrow G/A_n$ and F_n be as above. Clearly F_n is a normal subset of G so we may write it as a disjoint union of conjugacy classes, say $F_n = \bigcup C_{h_{n_j}}$. Note that each h_{n_j} is either in C_{g_0} or is not in C_{g_0} and is of the form $h_{nj} = g_0\varphi_{nj}$ with $\varphi_{nj} \in A_n$. Since the family (A_n) is descending, we have that $F_{n+1} \subset F_n$. Now suppose that an element $h = g_0\varphi$ appears in F_n and in F_{n+1} as a representative of a conjugacy class; then $\varphi \in A_{n+1}$. So if $h = g_0\varphi$ appears in every F_n then it follows that $\varphi \in \bigcap A_n = 1$. Hence $\bigcap F_n = C_{g_0}$. ∎

We still denote by Ψ_n the extension of $\Psi_n : G \longrightarrow G/A_n$ to the group rings $\mathbb{Z}G$ and $\mathbb{Z}G/A_n$. If $\alpha \in \mathcal{U}_1\mathbb{Z}G, g \in G$ then put $\beta_n = \Psi_n(\alpha)$ and $\overline{g} = \Psi_n(g)$.

Theorem 3.9 *Let G, A_n, α and β_n be as above. Given an element $g_0 \in G$ there exists $n_0 \in \mathbb{N}$ depending on g_0 such that $\widetilde{\beta}_{n_0}(\overline{g}_0) = \widetilde{\alpha}(g_0)$.*

Proof. Since α has finite support we can choose a finite number of elements of G, say, $g_1, \ldots, g_k$, representing the elements of the support of α. By Proposition 3.8, for every $1 \leq j \leq k$ there is an index m_j so that g_0 and g_j are not conjugate in G/A_{m_j}. Put $n_0 = max\{m_j\}$; then g_0 is not conjugate to g_j in G/A_{n_0} for every $1 \leq j \leq k$. It follows that $\widetilde{\beta}(\overline{g_0}) = \sum_{\overline{g} \sim \overline{g}_0} \alpha(g) = \widetilde{\alpha}(g_0)$. ∎

Corolarry 3.10 *Let G and (A_n) be as in the previous theorem. If each G/A_n is a UT-group then also is G.*

Proof. Since each G/A_n is a UT-group we have that $\alpha(g_0) \in \{0, 1\}$ and hence G is a UT-group. ∎

Theorem 3.11 *Let G be a polycyclic group and suppose that every finite quotient of G is UT-group. Then G is a UT-group.*

Proof. We use induction on the Hirsch length of G. It is clear that we may suppose that G is not finite. By [17], G contains an abelian normal torsion free subgroup A. Setting $A_n = A^n$ we obtain a descending chain. Since A is an abelian polycyclic group we have that $\bigcap A_n = 1$. Now, every G/A_n has shorter Hirsch length and since every finite quotient of G/A_n is isomorphic to a

finite quotient of G it follows that each G/A_n is a UT-group. The result follows by the previous corollary. ∎

Note that the theorem says that polycyclic groups are UT-groups if and only if every finite soluble group is a UT-group. So, the conjecture of Zassenhaus for finite groups would imply that every polycyclic group is a UT-group.

We now look at the p-UT property. Note that the former results also apply in this case.

Let $\mathcal{F}$ be a family of finite groups and G an arbitrary group. We say that G is an $\mathcal{F}$-group if every finite quotient of G is in $\mathcal{F}$. It is easy to see that every quotient of an $\mathcal{F}$-group is also an $\mathcal{F}$-group. Let we denote the set of finite soluble groups by $\mathcal{F}_s$ then it is clear that every polycyclic group is an $\mathcal{F}_s$-group. We denote also by $\mathcal{F}_f$, $\mathcal{F}_{ni}$, $\mathcal{F}_4$ respectively the families of finite Frobenius groups, groups with nilpotent derived subgroup and solvable groups whose order is not divisible by p^3, where p is any prime.

By results of [6, 11, 15] the families $\mathcal{F}_f,\ \mathcal{F}_{ni},\ \mathcal{F}_4$ all have the p-UT-property and so we have the following result.

Theorem 3.12 *Let G be a polycyclic group. If G is an $\mathcal{F}$-group, with $\mathcal{F}$ being one of the families above, then G is a p-UT group.*

Theorem 3.13 *Let G be a polycyclic group with nilpotent derived subgroup. Then G is a p-UT group. In particular supersoluble groups are p-UT groups.*

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