ACH2053 – Introdução à Estatística

Exercícios recomendados
Capítulo 2: Probabilidade Condicional

Fonte(Morris DeGroot, Mark Schervish. Probability and Statistics. 4th Ed.)

Seção 2.1:

Atenção: No livro do DeGroot, a notação “$\subset$” equivale a “$\subseteq$”

1. If $A \subset B$ with $\Pr(B) > 0$, what is the value of $\Pr(A|B)$?

2. If A and B are disjoint events and $\Pr(B) > 0$, what is the value of $\Pr(A|B)$?

3. If S is the sample space of an experiment and A is any event in that space, what is the value of $\Pr(A|S)$?

4. Each time a shopper purchases a tube of toothpaste, he chooses either brand A or brand B. Suppose that for each purchase after the first, the probability is 1/3 that he will choose the same brand that he chose on his preceding purchase and the probability is 2/3 that he will switch brands. If he is equally likely to choose either brand A or brand B on his first purchase, what is the probability that both his first and second purchases will be brand A and both his third and fourth purchases will be brand B?

5. A box contains r red balls and b blue balls. One ball is selected at random and its color is observed. The ball is then returned to the box and k additional balls of the same color are also put into the box. A second ball is then selected at random, its color is observed, and it is returned to the box together with k additional balls of the same color. Each time another ball is selected, the process is repeated. If four balls are selected, what is the probability that the first three balls will be red and the fourth ball will be blue?

6. A box contains three cards. One card is red on both sides, one card is green on both sides, and one card is red on one side and green on the other. One card is selected from the box at random, and the color on one side is observed. If this side is green, what is the probability that the other side of the card is also green?

9. Suppose that a box contains one blue card and four red cards, which are labeled A, B, C, and D. Suppose also that two of these five cards are selected at random, without replacement.

a. If it is known that card A has been selected, what is the probability that both cards are red?

b. If it is known that at least one red card has been selected, what is the probability that both cards are red?

13. A box contains three coins with a head on each side, four coins with a tail on each side, and two fair coins. If one of these nine coins is selected at random and tossed once, what is the probability that a head will be obtained?

14. A machine produces defective parts with three different probabilities depending on its state of repair. If the machine is in good working order, it produces defective parts with probability 0.02. If it is wearing down, it produces defective parts with probability 0.1. If it needs maintenance, it produces defective parts with probability 0.3. The probability that the machine is in good working order is 0.8, the probability that it is wearing down is 0.1, and the probability that it needs maintenance is 0.1. Compute the probability that a randomly selected part will be defective.

15. The percentages of voters classed as Liberals in three different election districts are divided as follows: in the first district, 21 percent; in the second district, 45 percent; and in the third district, 75 percent. If a district is selected at random and a voter is selected at random from that district, what is the probability that she will be a Liberal?

16. Consider again the shopper described in Exercise 4. On each purchase, the probability that he will choose the same brand of toothpaste that he chose on his preceding purchase is 1/3, and the probability that he will switch brands is 2/3. Suppose that on his first purchase the probability that he will choose brand A is 1/4 and the probability that he will choose brand B is 3/4. What is the probability that his second purchase will be brand B?

Seção 2.2:

1. If A and B are independent events and $\Pr(B) < 1$, what is the value of $\Pr(A^c|B^c)$?

2. Assuming that A and B are independent events, prove that the events A^c and B^c are also independent.

3. Suppose that A is an event such that $\Pr(A) = 0$ and that B is any other event. Prove that A and B are independent events.

4. Suppose that a person rolls two balanced dice three times in succession. Determine the probability that on each of the three rolls, the sum of the two numbers that appear will be 7.

5. Suppose that the probability that the control system used in a spaceship will malfunction on a given flight is 0.001. Suppose further that a duplicate, but completely independent, control system is also installed in the spaceship to take control in case the first system malfunctions. Determine the probability that the spaceship will be under the control of either the original system or the duplicate system on a given flight.

6. Suppose that 10,000 tickets are sold in one lottery and 5000 tickets are sold in another lottery. If a person owns 100 tickets in each lottery, what is the probability that she will win at least one first prize?

Dicas:

a) Na 1a loteria, a probabilidade de cada bilhete ser o ganhador é de 1/10.000, e na 2a loteria a probabilidade é de 1/5.000;
b) Lembre-se de que probabilidade de 1 ou mais ocorrências de um evento é igual a:
1 menos a probabilidade de 0 ocorrências do evento

7. Two students A and B are both registered for a certain course. Assume that student A attends class 80 percent of the time, student B attends class 60 percent of the time, and the absences of the two students are independent.

a. What is the probability that at least one of the two students will be in class on a given day?

b. If at least one of the two students is in class on a given day, what is the probability that A is in class that day?

8. If three balanced dice are rolled, what is the probability that all three numbers will be the same?

9. Consider an experiment in which a fair coin is tossed until a head is obtained for the first time. If this experiment is performed three times, what is the probability that exactly the same number of tosses will be required for each of the three performances?

10. The probability that any child in a certain family will have blue eyes is 1/4, and this feature is inherited independently by different children in the family. If there are five children in the family and it is known that at least one of these children has blue eyes, what is the probability that at least three of the children have blue eyes?

11. Consider the family with five children described in Exercise 10.

a. If it is known that the youngest child in the family has blue eyes, what is the probability that at least three of the children have blue eyes?

b. Explain why the answer in part (a) is different from the answer in Exercise 10.

12. Suppose that A, B, and C are three independent events such that $\Pr(A) = 1/4$, $\Pr(B) = 1/3$, and $\Pr(C) = 1/2$. **(a)** Determine the probability that none of these three events will occur. **(b)** Determine the probability that exactly one of these three events will occur.

13. Suppose that the probability that any particle emitted by a radioactive material will penetrate a certain shield is 0.01. If 10 particles are emitted, what is the probability that exactly one of the particles will penetrate the shield?

14. Consider again the conditions of Exercise 13. If 10 particles are emitted, what is the probability that at least one of the particles will penetrate the shield?

17. Two boys A and B throw a ball at a target. Suppose that the probability that boy A will hit the target on any throw is 1/3 and the probability that boy B will hit the target on any throw is 1/4. Suppose also that boy A throws first and the two boys take turns throwing. Determine the probability that the target will be hit for the first time on the third throw of boy A.

18. For the conditions of Exercise 17, determine the probability that boy A will hit the target before boy B does.

19. A box contains 20 red balls, 30 white balls, and 50 blue balls. Suppose that 10 balls are selected at random one at a time, with replacement; that is, each selected ball is replaced in the box before the next selection is made. Determine the probability that at least one color will be missing from the 10 selected balls.

Dica: Denote por R o evento de que pelo menos uma bola vermelha ser sorteada, e portanto R^c é o evento de que nenhuma bola vermelha foi sorteada; denote por W e W^c os eventos análogos para bolas brancas, e por B e B^c os eventos análogos para bolas azuis. Note que o evento de que pelo menos uma das bolas não foi sorteada nas 10 retiradas é $R^c \cup W^c \cup B^c$, cuja probabilidade $\Pr(R^c \cup W^c \cup B^c)$ pode ser calculada usando o Teorema 1.10.1 do livro (apresentado nos slides da aula T01 – Introdução à Probabilidade)

23. A programmer is about to attempt to compile a series of 11 similar programs. Let A_i be the event that the ith program compiles successfully for $i = 1, \ldots, 11$. When the programming task is easy, the programmer expects that 80 percent of programs should compile. When the programming task is difficult, she expects that only 40 percent of the programs will compile. Let B be the event that the programming task was easy. The programmer believes that the events $A_1, \ldots, A_{11}$ are conditionally independent given B and given B^c.

a. Compute the probability that exactly 8 out of 11 programs will compile given B.

b. Compute the probability that exactly 8 out of 11 programs will compile given B^c.

Dizer que A1…A11 são condicionalmente independents dado B significa que, se B ocorrer (a tarefa de programação é fácil), os eventos A1…A11 são independents, ou seja:

$$\Pr(A_1 \cap A_2 \cap \ldots \cap A_{11}|B) = \Pr(A_1|B)\ \Pr(A_2|B) \ldots \Pr(A_{11}|B).$$

Interpretação análoga vale para a afirmação de que A1…A11 são condicionalmente independents dado B^c.

Seção 2.3:

2. Consider again the conditions of Example 2.3.4 in this section, in which an item was selected at random from a batch of manufactured items and was found to be defective. For which values of i ($i = 1, 2, 3$) is the posterior probability that the item was produced by machine M_i larger than the prior probability that the item was produced by machine M_i?

Enunciado e solução do Exemplo 2.3.4 são apresentados nos slides da aula T02 – Probabilidade Condicional

3. Suppose that in Example 2.3.4 in this section, the item selected at random from the entire lot is found to be non-defective. Determine the posterior probability that it was produced by machine M_2.

4. A new test has been devised for detecting a particular type of cancer. If the test is applied to a person who has this type of cancer, the probability that the person will have a positive reaction is 0.95 and the probability that the person will have a negative reaction is 0.05. If the test is applied to a person who does not have this type of cancer, the probability that the person will have a positive reaction is 0.05 and the probability that the person will have a negative reaction is 0.95. Suppose that in the general population, one person out of every 100,000 people has this type of cancer. If a person selected at random has a positive reaction to the test, what is the probability that he has this type of cancer?

5. In a certain city, 30 percent of the people are Conservatives, 50 percent are Liberals, and 20 percent are Independents. Records show that in a particular election, 65 percent of the Conservatives voted, 82 percent of the Liberals voted, and 50 percent of the Independents voted. If a person in the city is selected at random and it is learned that she did not vote in the last election, what is the probability that she is a Liberal?

6. Suppose that when a machine is adjusted properly, 50 percent of the items produced by it are of high quality and the other 50 percent are of medium quality. Suppose, however, that the machine is improperly adjusted during 10 percent of the time and that, under these conditions, 25 percent of the items produced by it are of high quality and 75 percent are of medium quality.

a. Suppose that five items produced by the machine at a certain time are selected at random and inspected. If four of these items are of high quality and one item is of medium quality, what is the probability that the machine was adjusted properly at that time?

b. Suppose that one additional item, which was produced by the machine at the same time as the other five items, is selected and found to be of medium quality. What is the new posterior probability that the machine was adjusted properly?

7. Suppose that a box contains five coins and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the ith coin is tossed ($i = 1, \ldots, 5$), and suppose that $p_1 = 0$, $p_2 = 1/4$, $p_3 = 1/2$, $p_4 = 3/4$, and $p_5 = 1$.

a. Suppose that one coin is selected at random from the box and when it is tossed once, a head is obtained. What is the posterior probability that the ith coin was selected ($i = 1, \ldots, 5$)?

b. If the same coin were tossed again, what would be the probability of obtaining another head?

c. If a tail had been obtained on the first toss of the selected coin and the same coin were tossed again, what would be the probability of obtaining a head on the second toss?

8. Consider again the box containing the five different coins described in Exercise 7. Suppose that one coin is selected at random from the box and is tossed repeatedly until a head is obtained.

a. If the first head is obtained on the fourth toss, what is the posterior probability that the ith coin was selected ($i = 1, \ldots, 5$)?

b. If we continue to toss the same coin until another head is obtained, what is the probability that exactly three additional tosses will be required?

9. Consider again the conditions of Exercise 14 in Sec. 2.1. Suppose that several parts will be observed and that the different parts are conditionally independent given each of the three states of repair of the machine. If seven parts are observed and exactly one is defective, compute the posterior probabilities of the three states of repair.

14. Consider again the conditions of Exercise 23 in Sec. 2.2. Assume that $\Pr(B) = 0.4$. Let A be the event that exactly 8 out of 11 programs compiled. Compute the conditional probability of B given A.

Seção 2.5 (Exercícios suplementares):

2. Suppose that a fair coin is tossed repeatedly and independently until both a head and a tail have appeared at least once. **(a)** Describe the sample space of this experiment. **(b)** What is the probability that exactly three tosses will be required?

5. Suppose that in 10 rolls of a balanced die, the number 6 appeared exactly three times. What is the probability that the first three rolls each yielded the number 6?

8. Suppose that the events A and B are disjoint and that each has positive probability. Are A and B independent?

10. Suppose that each of two dice is loaded so that when either die is rolled, the probability that the number k will appear is 0.1 for $k = 1, 2, 5$, or 6 and is 0.3 for $k = 3$ or 4. If the two loaded dice are rolled independently, what is the probability that the sum of the two numbers that appear will be 7?

11. Suppose that there is a probability of 1/50 that you will win a certain game. If you play the game 50 times, independently, what is the probability that you will win at least once?

12. Suppose that a balanced die is rolled three times, and let X_i denote the number that appears on the ith roll ($i = 1, 2, 3$). Evaluate $\Pr(X_1 > X_2 > X_3)$.

13. Three students A, B, and C are enrolled in the same class. Suppose that A attends class 30 percent of the time, B attends class 50 percent of the time, and C attends class 80 percent of the time. If these students attend class independently of each other, what is (a) the probability that at least one of them will be in class on a particular day and (b) the probability that exactly one of them will be in class on a particular day?

15. Suppose that three red balls and three white balls are thrown at random into three boxes and and that all throws are independent. What is the probability that each box contains one red ball and one white ball?

Dica: Considere o espaço amostral como sendo todas as sequências de 6 dígitos, em que cada dígito pode ser 1,2 ou 3.
(Cada posição de uma sequência corresponde a uma das bolas, e o valor nessa posição corresponde à caixa em que a bola caiu).
Supor, sem perda de generalidade, que as 3 primeiras posições correspondem às bolas vermelhas, e as 3 últimas correspondem às bolas brancas.
P.ex. a sequencia 1 3 3 2 1 1 representa a situação em que as bolas vermelhas caíram nas caixas 1,3,3 respectivamente, e as bolas brancas caíram nas caixas 2,1,1 respectivamente.
Calcule o tamanho do espaço amostral e calcule a quantidade de sequências nas quais as 3 primeiras posições sejam permutações dos números {1,2,3} e as três últimas posições também sejam permutações dos números {1,2,3}. A razão entre essas duas grandezas fornecerá a probabilidade pedida.

23. Suppose that 80 percent of all statisticians are shy, whereas only 15 percent of all economists are shy. Suppose also that 90 percent of the people at a large gathering are economists and the other 10 percent are statisticians. If you meet a shy person at random at the gathering, what is the probability that the person is a statistician?

24. Dreamboat cars are produced at three different factories A, B, and C. Factory A produces 20 percent of the total output of Dreamboats, B produces 50 percent, and C produces 30 percent. However, 5 percent of the cars produced at A are lemons, 2 percent of those produced at B are lemons, and 10 percent of those produced at C are lemons. If you buy a Dreamboat and it turns out to be a lemon, what is the probability that it was produced at factory A?

25. Suppose that 30 percent of the bottles produced in a certain plant are defective. If a bottle is defective, the probability is 0.9 that an inspector will notice it and remove it from the filling line. If a bottle is not defective, the probability is 0.2 that the inspector will think that it is defective and remove it from the filling line.

a. If a bottle is removed from the filling line, what is the probability that it is defective?

b. If a customer buys a bottle that has not been removed from the filling line, what is the probability that it is defective?

27. Suppose that a family has exactly n children ($n \geq 2$). Assume that the probability that any child will be a girl is 1/2 and that all births are independent. Given that the family has at least one girl, determine the probability that the family has at least one boy.

29. Suppose that 13 cards are selected at random from a regular deck of 52 playing cards.

a. If it is known that at least one ace has been selected, what is the probability that at least two aces have been selected?

b. If it is known that the ace of hearts has been selected, what is the probability that at least two aces have been selected?

32. Consider again the conditions of Exercise 7 of Sec. 2.2. If exactly one of the two students A and B is in class on a given day, what is the probability that it is A?

33. Consider again the conditions of Exercise 2 of Sec. 1.10. If a family selected at random from the city subscribes to exactly one of the three newspapers A, B, and C, what is the probability that it is A?

34. Three prisoners A, B, and C on death row know that exactly two of them are going to be executed, but they do not know which two. Prisoner A knows that the jailer will not tell him whether or not he is going to be executed. He therefore asks the jailer to tell him the name of one prisoner other than A himself who will be executed. The jailer responds that B will be executed. Upon receiving this response, Prisoner A reasons as follows: Before he spoke to the jailer, the probability was 2/3 that he would be one of the two prisoners executed. After speaking to the jailer, he knows that either he or prisoner C will be the other one to be executed. Hence, the probability that he will be executed is now only 1/2. Thus, merely by asking the jailer his question, the prisoner reduced the probability that he would be executed from 2/3 to 1/2, because he could go through exactly this same reasoning regardless of which answer the jailer gave. Discuss what is wrong with prisoner A's reasoning.